

The equations underlying the perturbation analysis are a bit long and have been moved to this separate document, for that reason. They are no longer part of the QTM manual.

### 1.1 Perturbation analysis

The equations can be abbreviated by the use of the following variables:

Reduced mass:  $\mu_e = m_e/m_q$ ,  $\mu_f = m_f/m_q$

Reduced shear-wave impedance

$$\zeta_e(\omega) = \frac{Z_q^2}{\tilde{Z}_e^2(\omega)} - 1, \quad \zeta_f(\omega) = \frac{Z_q^2}{\tilde{Z}_f^2(\omega)} - 1, \quad \xi_{liq}(\omega) = \frac{\tilde{Z}_{liq}(\omega)}{Z_q} \quad \text{Eq. 1}$$

Index  $e$ : first layer (“electrode”)

Index  $f$ : second layer (“film”)

Index  $liq$ : bulk medium (“liquid”)

For brevity, the reduced shear-wave impedance was written without the tilde.

#### 1.1.1 Semi-infinite liquid

$$\begin{aligned} \text{SLA result: } \frac{\Delta \tilde{f}}{f_0} &= \frac{i}{\pi Z_q} \tilde{Z}_{liq} \\ \text{3rd order: } \frac{\Delta \tilde{f}}{f_{ref}} &= \frac{i}{n\pi} \left( \xi_{liq} + \frac{1}{3} \xi_{liq}^3 \right) \\ \text{5th order: } \frac{\Delta \tilde{f}}{f_{ref}} &= \frac{i}{n\pi} \left( \xi_{liq} + \frac{1}{3} \xi_{liq}^3 + \frac{1}{5} \xi_{liq}^5 \right) \end{aligned} \quad \text{Eq. 2}$$

#### 1.1.2 Viscoelastic film in air

$$\begin{aligned} \text{SLA result: } \frac{\Delta \tilde{f}}{f_0} &= \frac{-\tilde{Z}_f}{\pi Z_q} \tan(\tilde{k}_f d_f) \\ \text{3rd order: } \frac{\Delta \tilde{f}}{f_{ref}} &= -\mu_f + \mu_f^2 - \left( 1 + \frac{1}{3} (n\pi)^2 \zeta_f \right) \mu_f^3 \\ \text{5th order:} \\ \frac{\Delta \tilde{f}}{f_{ref}} &= -\mu_f + \mu_f^2 - \left( 1 + \frac{(n\pi)^2}{3} \zeta_f \right) \mu_f^3 + \left( 1 + \frac{4(n\pi)^2}{3} \zeta_f \right) \mu_f^4 \\ &\quad - \left( 1 + \frac{10(n\pi)^2}{3} \zeta_f + \frac{(n\pi)^4}{15} (1 - 2\zeta_f) \zeta_f \right) \mu_f^5 \end{aligned} \quad \text{Eq. 3}$$

### 1.1.3 Viscoelastic film in liquid

$$\text{SLA result: } \frac{\tilde{\Delta f}}{f_0} = \frac{-\tilde{Z}_f}{\pi Z_q} \frac{\tilde{Z}_f \tan(\tilde{k}_f d_f) - i\tilde{Z}_{liq}}{\tilde{Z}_f + i\tilde{Z}_{liq} \tan(\tilde{k}_f d_f)} - \frac{i\tilde{Z}_{liq}}{\pi Z_q} \quad \text{Eq. 4}$$

3rd order:

$$\begin{aligned} \frac{\tilde{\Delta f}}{f_{ref}} = & \frac{i}{n\pi} \left( \xi_{liq} + \frac{1}{3} \xi_{liq}^3 \right) - \left( 1 + \frac{i\xi_{liq}}{n\pi} - \zeta_f \xi_{liq}^2 \right) \mu_f \\ & + \left( 1 + \left( \frac{i}{n\pi} + in\pi\zeta_f \right) \xi_{liq} \right) \mu_f^2 - \left( 1 + \frac{(n\pi)^2}{3} \zeta_f \right) \mu_f^3 \end{aligned}$$

5th order

$$\begin{aligned} \frac{\tilde{\Delta f}}{f_{ref}} = & \frac{i\xi_{liq}}{n\pi} + \frac{i\xi_{liq}^3}{3n\pi} + \frac{i\xi_{liq}^5}{5n\pi} + \left( -1 - \frac{i\xi_{liq}}{n\pi} + \zeta_f \xi_{liq}^2 + \frac{(-5i + 15i\zeta_f)\xi_{liq}^3}{15n\pi} + \zeta_f \xi_{liq}^4 \right) \mu_f + \\ & \left( 1 + \left( \frac{i}{n\pi} + in\pi\zeta_f \right) \xi_{liq} - 4\zeta_f \xi_{liq}^2 + \frac{(5i - 45i\zeta_f)}{15n\pi} + \frac{(15i\pi^2\zeta_f - 15i\pi^2\zeta_f^2)}{15n\pi} \xi_{liq}^3 \right) \mu_f^2 + \\ & \left( -1 - \frac{1}{3}n^2\pi^2\zeta_f + \left( -\frac{i}{n\pi} - 4in\pi\zeta_f \right) \xi_{liq} + \left( 10\zeta_f + \frac{n^2(-10\pi^3\zeta_f + 20\zeta_f^2)}{15\pi} \right) \xi_{liq}^2 \right) \mu_f^3 + \\ & \left( 1 - \frac{4}{3}n^2\pi^2\zeta_f + \left( \frac{i}{n\pi} + 10in\pi\zeta_f + \frac{n^3(-5\pi^4\zeta_f + 10\pi^4\zeta_f^2)}{15\pi} \right) \xi_{liq} \right) \mu_f^4 + \\ & \left( -1 - \frac{10}{3}n^2\pi^2\zeta_f - \frac{1}{15}n^4\pi^4\zeta_f(-1 + 2\zeta_f) \right) \mu_f^5 \end{aligned}$$

### 1.1.4 Two viscoelastic films in air

SLA-Result:

$$\frac{\tilde{\Delta f}}{f_0} = \frac{-1}{\pi Z_q} \frac{\tilde{Z}_f \tan(\tilde{k}_f d_f) + \tilde{Z}_e \tan(\tilde{k}_e d_e)}{1 - \tilde{Z}_f / \tilde{Z}_e \tan(\tilde{k}_f d_f) \tan(\tilde{k}_e d_e)} \quad \text{Eq. 5}$$

3rd order:

$$\begin{aligned} \frac{\tilde{\Delta f}}{f_{ref}} = & -\mu_e + \mu_e^2 - \left( 1 + \frac{(n\pi)^2}{3} \zeta_e \right) \mu_e^3 - \left( 1 - 2\mu_e + 3 \left( 1 + \frac{(n\pi)^2}{3} \zeta_e \right) \mu_e^2 \right) \mu_f \\ & + \left( 1 - 3 \left( 1 + \frac{(n\pi)^2}{3} \zeta_e \right) \mu_e \right) \mu_f^2 - \left( 1 + \frac{(n\pi)^2}{3} \zeta_f \right) \mu_f^3 \end{aligned} \quad \text{Eq. 6}$$

5<sup>th</sup> order:

$$\begin{aligned} \frac{\Delta \tilde{f}}{f_{ref}} = & -\mu_e + \mu_e^2 + \left(-1 - \frac{1}{3}n^2\pi^2\zeta_e\right)\mu_e^3 + \left(1 + \frac{4}{3}n^2\pi^2\zeta_e\right)\mu_e^4 + \\ & \left(-1 - \frac{10}{3}n^2\pi^2\zeta_e - \frac{1}{15}n^4\pi^4\zeta_e(-1 + 2\zeta_e)\right)\mu_e^5 + \\ & \left(-1 + 2\zeta_e + (-3 - n^2\pi^2\zeta_e)\mu_e^2 + \left(4 + \frac{16}{3}n^2\pi^2\zeta_e\right)\mu_e^3 + \left(-5 - \frac{50}{3}n^2\pi^2\zeta_e - \frac{1}{3}n^4\pi^4\zeta_e(-1 + 2\zeta_e)\right)\mu_e^4\right)\mu_f + \\ & \left(1 + (-3 - n^2\pi^2\zeta_e)\mu_e + (6 + 8n^2\pi^2\zeta_e)\mu_e^2 + \left(-10 - \frac{100}{3}n^2\pi^2\zeta_e - \frac{2}{3}n^4\pi^4\zeta_e(-1 + 2\zeta_e)\right)\mu_e^3\right)\mu_f^2 + \\ & \left(-1 - \frac{1}{3}n^2\pi^2\zeta_f + \left(4 + \frac{4}{3}n^2\pi^2(3\zeta_e + \zeta_f)\right)\mu_e + \left(-10 - \frac{1}{3}n^4\pi^4\zeta_e(-2 + 3\zeta_e + \zeta_f)\right) - \frac{10}{3}n^2\pi^2(9\zeta_e + \zeta_f)\mu_e^2\right)\mu_f^3 + \\ & \left(1 + \frac{4}{3}n^2\pi^2\zeta_f + \left(-5 - \frac{1}{3}n^4\pi^4\zeta_e(-1 + 2\zeta_f) - \frac{10}{3}n^2\pi^2(3\zeta_e + 2\zeta_f)\right)\mu_e\right)\mu_f^4 + \\ & \left(-1 - \frac{10}{3}n^2\pi^2\zeta_f - \frac{1}{15}n^4\pi^4\zeta_f(-1 + 2\zeta_f)\right)\mu_f^5 \end{aligned} \quad \text{Eq. 7}$$

### 1.1.5 Two viscoelastic films in a liquid

SLA-Result:

$$\frac{\Delta \tilde{f}}{f_0} = \frac{-Z_e \tilde{Z}_f \left( \tilde{Z}_e \tan(\tilde{k}_e d_e) + \tilde{Z}_f \tan(\tilde{k}_f d_f) \right) + i \tilde{Z}_{liq} \left( \tilde{Z}_e \tan(\tilde{k}_f d_f) \tan(\tilde{k}_e d_e) - \tilde{Z}_f \right)}{\pi Z_q \tilde{Z}_f \left( \tilde{Z}_e - \tilde{Z}_f \tan(\tilde{k}_f d_f) \tan(\tilde{k}_e d_e) \right) + i \tilde{Z}_{liq} \left( \tilde{Z}_e \tan(\tilde{k}_f d_f) + \tilde{Z}_f \tan(\tilde{k}_e d_e) \right)} - \frac{i \tilde{Z}_{liq}}{\pi Z_q} \quad \text{Eq. 8}$$

3rd order:

$$\begin{aligned} \frac{\Delta \tilde{f}}{f_{ref}} = & \frac{i}{n\pi} \left( \xi_{liq} + \frac{1}{3}\xi_{liq}^3 \right) - \left( 1 + \frac{i}{n\pi} \xi_{liq} - \zeta_e \xi_{liq}^2 \right) \mu_e + \left( 1 + \left( \frac{i}{n\pi} + i n \pi \zeta_e \right) \xi_{liq} \right) \mu_e^2 \\ & - \left( 1 + \frac{1}{3}(n\pi)^2 \zeta_e \right) \mu_e^3 \\ & - \left( 1 + \frac{i}{n\pi} \xi_{liq} - \zeta_f \xi_{liq}^2 - 2 \left( 1 + \left( \frac{i}{n\pi} + i n \pi \zeta_e \right) \xi_{liq} \right) \mu_e + 3 \left( 1 + \frac{1}{3}(n\pi)^2 \zeta_e \right) \mu_e^2 \right) \mu_f \\ & + \left( 1 + \left( \frac{i}{n\pi} + i n \pi \zeta_f \right) \xi_e - 3 \left( 1 + \frac{1}{3}(n\pi)^2 \zeta_e \right) \mu_e \right) \mu_f^2 \\ & - \left( 1 + \frac{1}{3}(n\pi)^2 \zeta_f \right) \mu_f^3 \end{aligned} \quad \text{Eq. 9}$$

5th order:

Eq. 10

$$\begin{aligned}
\frac{\Delta \tilde{f}}{f_{ref}} = & \frac{i \xi_{liq}}{n\pi} + \frac{i \xi_{liq}^3}{3n\pi} + \frac{i \xi_{liq}^5}{5n\pi} + \left( -1 - \frac{i \xi_{liq}}{n\pi} + \zeta_e \xi_{liq}^2 + \frac{(-5i + 15i \zeta_e) \xi_{liq}^3}{15n\pi} + \zeta_e \xi_{liq}^4 \right) \mu_f + \\
& \left( 1 + \left( \frac{i}{n\pi} + i n \pi \zeta_e \right) \xi_{liq} - 4 \zeta_e \xi_{liq}^2 + \left( \frac{5i - 45i \zeta_e}{15n\pi} + \frac{n(15i \pi^2 \zeta_e - 15i \pi^2 \zeta_e^2)}{15n\pi} \right) \xi_{liq}^3 \right) \mu_e^2 + \\
& \left( -1 - \frac{1}{3} n^2 \pi^2 \zeta_e + \left( -\frac{i}{n\pi} - 4i n \pi \zeta_e \right) \xi_{liq} + \left( 10 \zeta_e + \frac{n^2(-10\pi^3 \zeta_e + 20\pi^3 \zeta_e^2)}{15\pi} \right) \xi_{liq}^2 \right) \mu_e^3 + \\
& \left( 1 - \frac{4}{3} n^2 \pi^2 \zeta_e + \left( \frac{i}{n\pi} + 10i n \pi \zeta_e + \frac{1}{3} i n^3 \pi^3 \zeta_e (-1 + 2\zeta_e) \right) \xi_{liq} \right) \mu_e^4 + \\
& \left( -1 - \frac{10}{3} n^2 \pi^2 \zeta_e - \frac{1}{15} n^4 \pi^4 \zeta_e (-1 + 2\zeta_e) \right) \mu_f^5 + \\
& \left( -1 - \frac{i \xi_{liq}}{n\pi} + \zeta_f \xi_{liq}^2 + \frac{(-5i + 15i \zeta_f) \xi_{liq}^3}{15n\pi} + \zeta_f \xi_{liq}^4 + \right. \\
& \quad \left( 2 + \left( \frac{2i}{n\pi} + 2i n \pi \zeta_e \right) \xi_{liq} + \frac{(-90\pi \zeta_e - 30\pi \zeta_f) \xi_{liq}^2}{15\pi} + \left( \frac{10i - 60i \zeta_e - 30i \zeta_f}{15n\pi} - 2i n \pi \zeta_e (-1 + \zeta_f) \right) \xi_{liq}^3 \right) \mu_e + \\
& \quad \left( -3n^2 \pi^2 \zeta_e + \left( -\frac{3i}{n\pi} - 12i n \pi \zeta_e \right) \xi_{liq} + \left( \frac{n^2(45\pi^3 \zeta_e^2 + 15\pi^2 \zeta_e (-2 + \zeta_f))}{15\pi} + \frac{405\pi \zeta_e + 45\zeta_f}{15\pi} \right) \xi_{liq}^2 \right) \mu_e^2 + \\
& \quad \left( 4 + \frac{16}{3} n^2 \pi^2 \zeta_e + \left( \frac{4i}{n\pi} + 40i n \pi \zeta_e + \frac{4}{3} i n^3 \pi^3 \zeta_e (-1 + 2\zeta_e) \right) \xi_{liq} \right) \mu_e^3 + \left( -5 - \frac{50}{3} n^2 \pi^2 \zeta_e - \frac{1}{3} n^4 \pi^4 \zeta_e (-1 + 2\zeta_e) \right) \mu_e^4 \Big) \mu_f + \\
& \left( 1 + \left( -\frac{i}{n\pi} + i n \pi \zeta_f \right) \xi_{liq} - 4 \zeta_f \xi_{liq}^2 + \left( \frac{5i - 45i \zeta_f}{15n\pi} + \frac{n(15i \pi^2 \zeta_f - 15i \pi^2 \zeta_f^2)}{15\pi} \right) \xi_{liq}^3 + \right. \\
& \quad \left( -3 - n^2 \pi^2 \zeta_e + \left( -\frac{3i}{n\pi} - 3i n \pi \zeta_e (3\zeta_e + \zeta_f) \right) \xi_{liq} + 2n^2 \pi^2 \zeta_e (-1 + 2\zeta_f) + \frac{270\pi \zeta_e + 180\zeta_f}{15\pi} \right) \xi_{liq}^2 \Big) \mu_e + \\
& \quad \left( 6 + 8n^2 \pi^2 \zeta_e + \left( \frac{6i}{n\pi} + i n^3 \pi^3 \zeta_e (-2 + 3\zeta_e + \zeta_f) + \frac{n(810i \pi^2 \zeta_e + 90i \pi^2 \zeta_f^2)}{15\pi} \right) \xi_{liq} \right) \mu_e^2 + \\
& \quad \left( -10 + \frac{100}{3} n^2 \pi^2 \zeta_e + \frac{2}{3} n^4 \pi^4 \zeta_e (-1 + 2\zeta_e) \right) \mu_e^3 \Big) \mu_f^2 + \\
& \left( -1 - \frac{1}{3} n^2 \pi^2 \zeta_f + \left( -\frac{i}{n\pi} - 4i n \pi \zeta_f \right) \xi_{liq} + \left( 10 \zeta_f + \frac{n^2(-10\pi^3 \zeta_f + 20\pi^3 \zeta_f^2)}{15\pi} \right) \xi_{liq}^2 + \right. \\
& \quad \left( 1 - \frac{4}{3} n^2 \pi^2 (3\zeta_e + \zeta_f) + \left( \frac{4i}{n\pi} + \frac{4}{3} i n^3 \pi^3 \zeta_e (-1 + 2\zeta_f) + 8i n \pi (3\zeta_e + 2\zeta_f) \right) \xi_{liq} \right) \mu_e + \\
& \quad \left( -10 + \frac{1}{3} n^4 \pi^4 \zeta_e (-2 + 3\zeta_e + 2\zeta_f) - \frac{10}{3} n^2 \pi^2 (9\zeta_e + \zeta_f) \right) \mu_e^2 \Big) \mu_f^3 + \\
& \left( 1 + \frac{4}{3} n^2 \pi^2 \zeta_f + \left( \frac{i}{n\pi} + 10i n \pi \zeta_f + \frac{n^2(-5i n \pi^4 \zeta_f + 10\pi^4 \zeta_f^2)}{15\pi} \right) \xi_{liq} + \left( 5 - \frac{1}{3} n^4 \pi^4 \zeta_e (-1 + 2\zeta_f) - \frac{10}{3} n^2 \pi^2 (3\zeta_e + 2\zeta_f) \right) \mu_e \right) \mu_f^4 + \\
& \left( 1 - \frac{10}{3} n^2 \pi^2 \zeta_f - \frac{1}{15} n^4 \pi^4 (-1 + 2\zeta_f) \right) \mu_f^5
\end{aligned}$$